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1983 J. Phys. A: Math. Gen. 16 2353

(http://iopscience.iop.org/0305-4470/16/11/009)

J. Phys. A: Math. Gen. 16 (1983) 2353–2361. Printed in Great Britain

On the conditions for definiteness of energy and charge

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Received 11 May 1981, in final form 29 March 1983

Abstract. The algebraic conditions of definiteness of energy and charge for the first-order wave equations are analysed and re-examined. The known definiteness conditions are applicable when there is no mass–spin degeneracy. A procedure is suggested for their use in the degenerate case, and is illustrated by application to an indefinite theory of particles of spin $\frac{3}{2}$ and $\frac{1}{2}$.

1. Introduction

Conditions of definiteness of energy and charge are required when we deal with first-order wave equations. By definiteness conditions we mean algebraic conditions which enable us to determine the definiteness of a given equation from its β algebra. It should be mentioned that we have no universal algebraic definiteness conditions; the existing ones are applicable only under certain conditions.

Recently the definiteness problem was treated by Amar and Dozzio (1972), Cox (1974a, b), Fedorov and Pletyukhov (1974) and Loide and Loide (1977). In Amar and Dozzio (1972) it was proved that for definiteness it is sufficient that to a given spin s there corresponds not more than one particle. As is shown by analysing particular equations (Cox 1974a, b, Loide and Loide 1977), in the many-particle case the equations are mostly indefinite; the densities of energy or charge of different particles have different signs. But there are also definite many-particle equations, as for example the equation for spin-2 and spin-0 particles given by Cox (1974b).

The algebraic definiteness conditions were first derived by Fedorov (1958) assuming that the equation describes not more than one particle, and are not applicable in the many-particle case, as was also shown in Loide and Loide (1977). Fedorov and Pletyukhov (1974) used Fedorov's definiteness conditions and essentially proved that in the case of first-order single-particle equations definiteness is always satisfied.

The definiteness conditions given by Fedorov (1958) were applied separately to each mass–spin state in (Cox 1974a) assuming that there is no mass–spin degeneracy, i.e. to each mass–spin state (m, s) corresponds only one particle. These conditions do not apply when states of the same mass and spin occur.

In this paper we demonstrate how to use the algebraic definiteness conditions in the case of mass–spin degeneracy. Our idea is the following: we reduce the given equation to a new one which has no mass–spin degeneracy and which in the limit reduces to the former one. The new equation has no mass–spin degeneracy and therefore allows us to use the existing definiteness conditions. Thus the algebraic definiteness conditions may be regarded as the most universal definiteness conditions.

The paper is organised as follows. In § 2 the formalism of spin-projection operators is introduced. In §§ 3 and 4 the known algebraic definiteness conditions are analysed in general and single mass cases. At the same time some mistakes of Loide and Loide (1977) are corrected. In § 5 a method (the ε procedure) is suggested for testing definiteness in the degenerate case. In §§ 6 and 7 we illustrate the usage of definiteness conditions on equations which describe spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ particles.

2. General formalism

We deal with the first-order wave equations

$$(p_\mu\beta^\mu - \varkappa)\psi(p) = 0 \tag{1}$$

where $\varkappa > 0$ and $\psi(p)$ transforms according to some finite-dimensional representation of the Lorentz group. As is known (Corson 1953, Gel'fand *et al* 1963), all non-zero eigenvalues $\pm b$ of β^0 are connected with the masses by the relation $m = \varkappa/b$.

In the following we use the formalism based on the spin projection operators and treated by Loide (1972), Loide and Loide (1977) and Biritz (1979). This formalism was previously used by Weinberg (1964a, b, 1969), Pursey (1965) and Tung (1966, 1967).

We may restrict ourselves to the rest system. Then all the useful information is contained in the β^0 matrix. It is useful to represent β^0 in the form

$$\beta^0 = \beta^{(s_1)} + \beta^{(s_2)} + \ldots + \beta^{(s_n)}, \tag{2}$$

where $\beta^{(s_i)}$ includes only spin projection operators of spin s_i (Loide 1974, Loide and Loide 1977). $\beta^{(s)}$ are not in general projection operators. Our matrices $\beta^{(s)}$ are essentially Gel'fand–Yaglom spin-blocks inflated to the size of β^0 by rows and columns of zeros.

When our equation describes particles with spin s the corresponding $\beta \equiv \beta^{(s)}$ has non-zero eigenvalues $\pm b_i$ and satisfies the minimal equation (minimal polynomial)

$$\beta^a(\beta^2 - b_1^2)\ldots(\beta^2 - b_k^2) = 0, \tag{3}$$

where $a \geqslant 0$. When our equation does not describe the particle with spin s, the corresponding $\beta \equiv \beta^{(s)}$ has only eigenvalues equal to zero and β is nilpotent,

$$\beta^a = 0, \tag{4}$$

where $a \geqslant 2$.

The solutions which describe particles with mass $m = \varkappa/b$ and spin s are given in the form

$$\psi^s_{\pm m} = P^s_{\pm b}\psi \tag{5}$$

(+ describes particles, − antiparticles), where $P^s_{\pm b}$ is the corresponding projection operator.

In the case of the minimal polynomial (3), $P^s_{\pm b}$ is given by the expression ($\beta \equiv \beta^{(s)}$)

$$P^s_{\pm b_i} = (\pm 1)^{a+1} A\beta^a(\beta^2 - b_1^2)\ldots(\beta^2 - b_{i-1}^2)(\beta \pm b_i)(\beta^2 - b_{i+1}^2)\ldots(\beta^2 - b_k^2), \tag{6}$$

where

$$A^{-1} = 2b_i^{a+1}(b_i^2 - b_1^2)\ldots(b_i^2 - b_{i-1}^2)(b_i^2 - b_{i+1}^2)\ldots(b_i^2 - b_k^2). \tag{7}$$

3. Algebraic definiteness conditions

In this section we discuss the definiteness conditions needed in the following sections and clarify the assumptions when these conditions are applicable.

The definiteness of energy means the definiteness of the scalar product

$$(\psi, \psi) = \psi^+ \Lambda \psi \tag{8}$$

for each solution of the wave equation (Gel'fand *et al* 1963). Λ is the hermitising matrix of the scalar product. The definiteness of charge demands the definiteness of

$$(\psi, \beta^0 \psi) = \psi^+ \Lambda \beta^0 \psi. \tag{9}$$

The conditions (8) and (9) are expressible by the help of projection operators $P^s_{\pm b}$ in the form (Fedorov 1958, Cox 1974a, Loide and Loide 1977)

$$(\psi^s_{\pm m}, \psi^s_{\pm m}) = \psi^+ \Lambda P^s_{\pm b} \psi, \tag{10}$$

$$(\psi^s_{\pm m}, \beta^0 \psi^s_{\pm m}) = \pm \psi^+ \Lambda P^s_{\pm b} \psi. \tag{11}$$

As we see, we have to deal with the expression $\psi^+ \Lambda P^s_{\pm b} \psi$ and we are interested in its sign.

Assuming that P^s_b has only one non-zero eigenvector (we do not take into consideration degeneracy due to spin projection), the following condition is valid (Fedorov 1958):

$$\mathrm{sgn}(\psi^+ \Lambda P^s_{\pm b} \psi) = \mathrm{sgn}\, \mathrm{Tr}(\Lambda P^s_{\pm b}). \tag{12}$$

From (12) the investigation of definiteness reduces to the investigation of the sign of $\mathrm{Tr}(\Lambda P^s_{\pm b})$. Due to (6), $P^s_{\pm b}$ is expressed as a polynomial of $\beta \equiv \beta^{(s)}$,

$$P^s_{\pm b} = \sum_{j=a}^{2k+a-1} \alpha_j \beta^j, \tag{13}$$

where α_j are some coefficients. Therefore $\mathrm{Tr}(\Lambda P^s_{\pm b})$ is the following:

$$\mathrm{Tr}(\Lambda P^s_{\pm b}) = \sum_{j=a}^{2k+a-1} \alpha_j \, \mathrm{Tr}(\Lambda \beta^j). \tag{14}$$

As we see the investigation of $\mathrm{Tr}(\Lambda P^s_{\pm b})$ reduces to the investigation of expressions $\mathrm{Tr}(\Lambda \beta^j)$. In Cox (1974a), Fedorov and Pletyukhov (1974) it is shown that for integer spin

$$\mathrm{Tr}(\Lambda \beta^{2j+1}) = 0, \qquad \mathrm{Tr}(\Lambda \beta^{2j}) \neq 0, \tag{15}$$

and for half-odd-integer spin

$$\mathrm{Tr}(\Lambda \beta^{2j+1}) \neq 0, \qquad \mathrm{Tr}(\Lambda \beta^{2j}) = 0. \tag{16}$$

Therefore in (14) we must deal with even powers of β in the former case and with odd powers of β in the latter case.

The relation (12) is fundamental to the algebraic definiteness testing. Equation (12) holds only if there is no mass–spin degeneracy, i.e. for a given eigenvalue b there exists only one spin-s state. Therefore one more condition is useful in considering the definiteness problem. The number of particles in our equation with given mass and spin is obtained from the expression (Loide and Loide 1977)

$$\mathrm{Tr}\, P^s_b = N(2s+1). \tag{17}$$

Here N is the number of particles. When $N=1$ the trace condition (12) is valid and we may use it to determine the definiteness. In the case when $N>1$, i.e. the equation describes several particles with the same mass and spin, the trace condition (12) may not be valid and may lead to incorrect results (Loide and Loide 1977).

4. Single mass equations

Let us write down the definiteness conditions for the single mass equations having more than one spin. Now β^0 has only two non-zero eigenvalues $+b$ and $-b$, and without loss of generality we may set $b=1$. The minimal polynomial (3) takes the form

$$\beta^a(\beta^2-1)=0 \tag{18}$$

and $P^s_{\pm 1}$ is

$$P^s_{\pm 1}=(\pm 1)^{a+1}\tfrac{1}{2}\beta^a(\beta\pm 1). \tag{19}$$

The expression (14) takes the form

$$\mathrm{Tr}(\Lambda P^s_{\pm 1})=(\pm 1)^a\tfrac{1}{2}[\mathrm{Tr}(\Lambda\beta^{a+1})\pm\mathrm{Tr}(\Lambda\beta^a)]. \tag{20}$$

Using relations (15) and (16) we get:

(a) integer spin:

$$\mathrm{Tr}(\Lambda P^s_{\pm 1})=\gamma_s; \tag{21}$$

(b) half-odd-integer spin:

$$\mathrm{Tr}(\Lambda P^s_{\pm 1})=\pm\gamma_s \tag{22}$$

where $2\gamma_s=\mathrm{Tr}(\Lambda\beta^{a+1})$ or $2\gamma_s=\mathrm{Tr}(\Lambda\beta^a)$ is the non-zero trace.

Now the trace condition (12) gives from (21) and (22) the following definiteness conditions:

(a) integer spin:

$$\begin{aligned} &\text{sign of energy} && \mathrm{sgn}(\psi^s_\pm,\psi^s_\pm)=\mathrm{sgn}\,\gamma_s,\\ &\text{sign of charge} && \mathrm{sgn}(\psi^s_\pm,\beta^0\psi^s_\pm)=\pm\mathrm{sgn}\,\gamma_s; \end{aligned} \tag{23}$$

(b) half-odd-integer spin:

$$\begin{aligned} &\text{sign of energy} && \mathrm{sgn}(\psi^s_\pm,\psi^s_\pm)=\pm\mathrm{sgn}\,\gamma_s,\\ &\text{sign of charge} && \mathrm{sgn}(\psi^s_\pm,\beta^0\psi^s_\pm)=\mathrm{sgn}\,\gamma_s. \end{aligned} \tag{24}$$

As we see from (23) and (24), the definiteness is satisfied if $\gamma_s\neq 0$ and if the different γ_s have the same sign. From (23) and (24) we also find that in the single-particle case the energy is definite for bosons and the charge is definite for fermions (Fedorov and Pletyukhov 1974).

The definiteness conditions given in Fedorov (1958) were derived for β^0 assuming that the relation (12) is satisfied, i.e. assuming that there exists only one mass–spin state. In that case we may operate with β^0 instead of with $\beta\equiv\beta^{(s)}$. The definiteness conditions are written in the form (Fedorov 1958):

(a) definiteness of energy:

$$(-1)^{a+1}\{[\mathrm{Tr}(\Lambda(\beta^0)^{a+1})]^2-[\mathrm{Tr}(\Lambda(\beta^0)^a)]^2\}>0; \tag{25}$$

(b) definiteness of charge:

$$(-1)^a\{[\mathrm{Tr}(\Lambda(\beta^0)^{a+1})]^2-[\mathrm{Tr}(\Lambda(\beta^0)^a)]^2\}>0. \tag{26}$$

Now a is the degree in the minimal polynomial of β^0. In the single-particle case the conditions (25) and (26) are equivalent to (23) and (24) because one of the traces in brackets is always equal to zero.

In § 5 we demonstrate how to use the trace conditions in the degenerate case. The idea is the following: we replace the given equation by a new one which has no mass–spin degeneracy and can therefore be tested for definiteness using (12).

5. Degenerate case

Let us suppose that the wave equation describes several particles with given mass and spin. For the sake of simplicity we also suppose that in the case of a given spin s the corresponding particles have the same mass m. Then $\beta\equiv\beta^{(s)}$ satisfies the minimal equation (we also take $b=1$)

$$\beta^a(\beta^2-1)=0. \tag{27}$$

If there are k particles with the same mass and spin we consider a new equation which describes particles with the same spin but with k different masses: m, $m/(1+\varepsilon_1),\ldots,m/(1+\varepsilon_{k-1})$. Then the eigenvalues of $\beta^{(s)}\equiv\beta_\varepsilon$ are ±1, $\pm(1+\varepsilon_1),\ldots,\pm(1+\varepsilon_{k-1})$ and β_ε satisfies instead of (27) the minimal equation

$$\beta_\varepsilon^a(\beta_\varepsilon^2-1)[\beta_\varepsilon^2-(1+\varepsilon_1)^2]\ldots[\beta_\varepsilon^2-(1+\varepsilon_{k-1})^2]=0. \tag{28}$$

Also we demand that in the limit $\varepsilon_i\to0$ we get β which satisfies (27). Due to different masses we may use the trace condition (12) and determine the definiteness of different solutions. After this is done we take the limit $\varepsilon_i\to0$. Since in the final stage we are interested in the limit $\varepsilon_i\to0$ we may suppose at the beginning that ε_i are infinitesimal parameters.

This ε procedure is performed in the following way. A general first-order wave equation always contains free parameters which allow us to vary the eigenvalues of the β^0 matrix and also the eigenvalues of $\beta^{(s)}$ (Loide 1972, Loide and Loide 1977). If we have, for example, the minimal equation (27) we may infinitesimally vary the parameters and in this way infinitesimally alter the eigenvalues of $\beta^{(s)}$. In the limit, $\lim_{\varepsilon\to0}\beta_\varepsilon=\beta$, where β satisfies (27). The given procedure is in principle possible in the case of an arbitrary first-order wave equation. As we see in § 7 this ε procedure is not unique, but in definiteness testing this is not necessary.

So, introducing the ε procedure, we demanded that there exists the matrix β_ε which removes the degeneracy and in the limit, $\lim_{\varepsilon\to0}\beta_\varepsilon=\beta$, where β satisfies (27). We think the ε procedure is useful in the theory of relativistic wave equations in the degenerate case. It also serves the possibility of separating the degenerate states.

In the following sections we deal with the single mass equations which describe spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ particles and illustrate how to determine the definiteness and perform the ε procedure. The representation we have used serves as the simplest non-trivial example with degenerate states.

6. Spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ equations

We deal with equations with the representation $(1,\frac{1}{2})\oplus(0,\frac{1}{2})\oplus(\frac{1}{2},0)\oplus(\frac{1}{2},1)$ which describe particles with spins $\frac{3}{2}$ and $\frac{1}{2}$. Now β^0 has the form $\beta^0=\beta^{(3/2)}+\beta^{(1/2)}$. Denoting the representations $1=(1,\frac{1}{2})$, $2=(0,\frac{1}{2})$, $3=(\frac{1}{2},0)$ and $4=(\frac{1}{2},1)$, the matrices $\beta^{(3/2)}$, $\beta^{(1/2)}$ and Λ are the following (Loide and Loide 1977):

$$\beta^{(3/2)}=\begin{vmatrix} 0 & 0 & 0 & t_{14}^{(3/2)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t_{41}^{(3/2)} & 0 & 0 & 0 \end{vmatrix}, \tag{29}$$

$$\beta^{(1/2)}=\begin{vmatrix} 0 & 0 & at_{13}^{(1/2)} & \frac{1}{2}t_{14}^{(1/2)} \\ 0 & 0 & ct_{23}^{(1/2)} & bt_{24}^{(1/2)} \\ bt_{31}^{(1/2)} & ct_{32}^{(1/2)} & 0 & 0 \\ \frac{1}{2}t_{41}^{(1/2)} & at_{42}^{(1/2)} & 0 & 0 \end{vmatrix}, \tag{30}$$

$$\Lambda=\begin{vmatrix} 0 & 0 & 0 & \rho_1(t_{14}^{(3/2)}-t_{14}^{(1/2)}) \\ 0 & 0 & \rho_2 t_{23}^{(1/2)} & 0 \\ 0 & \rho_2 t_{32}^{(1/2)} & 0 & 0 \\ \rho_1(t_{41}^{(3/2)}-t_{41}^{(1/2)}) & 0 & 0 & 0 \end{vmatrix}, \tag{31}$$

where $t_{ij}^{(s)}$ are spin projection operators satisfying

$$t_{ij}^{(s)}t_{jk}^{(s')}=t_{ik}^{(s)}\delta_{ss'}. \tag{32}$$

The relation (32) is needed when operating with matrices β and Λ.

If we deal with the single mass equations we have four different choices of coefficients a, b, c, ρ_1 and ρ_2 (Loide 1974, Loide and Loide 1977).

$$\text{Case I:}\qquad ab=-\tfrac{1}{4},\qquad c=-\tfrac{1}{2},\qquad \rho_1=\rho_2. \tag{33}$$

$$\text{Case II:}\qquad ab=\tfrac{1}{4},\qquad c=\tfrac{1}{2},\qquad \rho_1=-\rho_2. \tag{34}$$

$$\text{Case III:}\qquad ab=-\tfrac{3}{4},\qquad c=-\tfrac{3}{2},\qquad \rho_1=\rho_2. \tag{35}$$

$$\text{Case IV:}\qquad ab=\tfrac{3}{4},\qquad c=-\tfrac{1}{2},\qquad \rho_1=-\rho_2. \tag{36}$$

In case I the equation describes a single spin-$\frac{3}{2}$ particle and is the Pauli–Fierz $s=\frac{3}{2}$ equation (which in turn is equivalent to the Rarita–Schwinger equation). $\beta^{(3/2)}$ satisfies the minimal equation (18) with $a=1$, and $\beta^{(1/2)}$ is nilpotent and satisfies (4) with $a=2$.

In cases II and III the equation describes one spin-$\frac{3}{2}$ particle and one spin-$\frac{1}{2}$ particle. The minimal equation of $\beta^{(3/2)}$ is the same as in the Pauli–Fierz case; $\beta^{(1/2)}$ satisfies the minimal equation (18) with $a=1$.

In case IV the equation describes one spin-$\frac{3}{2}$ and two spin-$\frac{1}{2}$ particles. The minimal equations of $\beta^{(3/2)}$ and $\beta^{(1/2)}$ are the same as in cases II and III but β^0 satisfies $(\beta^0)^2=I$. In this case our equation is equivalent to the Dirac equation for a vector bispinor.

Now we examine the definiteness of charge. In all four cases we get the same result for $\gamma_{3/2}$ because $\beta^{(3/2)}$ satisfies the minimal equation (18) with $a=1$ and we get

$$\gamma_{3/2}=4\rho_1 \tag{37}$$

(in calculating traces, the relation $\mathrm{Tr}\, t_{ii}^{(s)}=2s+1$ must be used).

In cases II and III the equation describes in addition to a spin-$\frac{3}{2}$ particle also one spin-$\frac{1}{2}$ particle. For $\gamma_{1/2}$ we get

$$\text{case II} \qquad \gamma_{1/2} = -2\rho_1, \tag{38}$$

$$\text{case III} \qquad \gamma_{1/2} = -4\rho_1. \tag{39}$$

From (37)–(39) it is obvious that charge densities of spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ particles have opposite signs.

In case IV (two spin-$\frac{1}{2}$ particles) the trace condition (12) is not satisfied, and our previous definiteness conditions give the wrong result. From (30), (31) and (36) we get for $\gamma_{1/2}$

$$\gamma_{1/2} = 0. \tag{40}$$

As we see in § 7, charge densities of different spin-$\frac{1}{2}$ particles have opposite signs. Therefore the result $\gamma_s = 0$ for some spin s may indicate the indefiniteness of states of the same mass and spin.

Concerning the definiteness conditions (26), it is possible to verify that these conditions indeed work only in the single-particle case.

7. ε procedure

Now we deal with case IV, when our equation describes in addition to a spin-$\frac{3}{2}$ particle two spin-$\frac{1}{2}$ particles. To use the trace condition (12) we find the new equation which describes two spin-$\frac{1}{2}$ particles with masses m and $m/1+\varepsilon$ and which in the limit $\varepsilon \to 0$ reduces to the original equation.

Since we are interested in the definiteness of spin-$\frac{1}{2}$ particles we deal with the matrix $\beta \equiv \beta^{(1/2)}$ only. In case IV β satisfies the minimal equation

$$\beta(\beta^2 - 1) = 0. \tag{41}$$

From (30), $\beta^{(1/2)}$ contains three parameters a, b and c, but eigenvalues depend on two parameters ab and c. Having two parameters, it is possible to vary the eigenvalues of $\beta^{(1/2)}$. In (41), ab and c are given by (36). If we vary them infinitesimally

$$ab \to \tfrac{3}{4} + \delta_1, \qquad c \to -\tfrac{1}{2} + \delta_2, \tag{42}$$

the eigenvalues of $\beta^{(1/2)}$ are changed.

A new β_ε matrix (cf § 5) must have the non-zero eigenvalues ± 1 and $\pm(1+\varepsilon)$, and satisfy the minimal equation

$$\beta_\varepsilon(\beta_\varepsilon^2 - 1)[\beta_\varepsilon^2 - (1+\varepsilon)^2] = 0. \tag{43}$$

The characteristic equation of β is from (30) and (32)

$$\mathrm{Det}|\beta - \lambda| = \lambda^5 - (2ab + c^2 + \tfrac{1}{4})\lambda^3 + [ab(ab - c) + c^2/4]\lambda = 0. \tag{44}$$

Demanding that β_ε satisfies (43), we get from (44) the following system to determine the new parameters ab_ε and c_ε:

$$2ab_\varepsilon + c_\varepsilon^2 + \tfrac{1}{4} = 2 + 2\varepsilon, \qquad ab_\varepsilon(ab_\varepsilon - c_\varepsilon) + c_\varepsilon^2/4 = 1 + 2\varepsilon. \tag{45}$$

Since we are interested in the limit $\varepsilon \to 0$ we retain only the terms linear in ε in (45).

Substituting (42) into (45) we get

$$2\delta_1 - \delta_2 = 2\varepsilon. \tag{46}$$

From (46) the choice of δ_1 and δ_2 is not unique. This non-uniqueness is caused by the fact that at an infinitesimal level the system (45) is not sensitive with regard to (43). In other words, demanding the eigenvalues $\pm(1+\varepsilon_1)$ and $\pm(1+\varepsilon_2)$ the system (45) remains the same. It is possible to verify that such different choices do not affect the results of definiteness testing.

In (46) we choose

$$\delta_1 = \varepsilon/2, \qquad \delta_2 = -\varepsilon;$$

then

$$ab_\varepsilon = \tfrac{3}{4} + \varepsilon/2, \qquad c_\varepsilon = -\tfrac{1}{2} - \varepsilon. \tag{47}$$

Evidently as $\varepsilon \to 0$, $\beta_\varepsilon \to \beta$.

Let us write down the projection operators and determine the definiteness of charge. From (6) we get

$$\begin{aligned} P^{1/2}_{\pm 1} &= -(4\varepsilon)^{-1}\beta_\varepsilon(\beta_\varepsilon \pm 1)(\beta_\varepsilon^2 - 1 - 2\varepsilon), \\ P^{1/2}_{\pm(1+\varepsilon)} &= (4\varepsilon)^{-1}\beta_\varepsilon(\beta_\varepsilon^2 - 1)(\beta_\varepsilon \pm 1 \pm \varepsilon). \end{aligned} \tag{48}$$

Now we calculate $\mathrm{Tr}(\Lambda P^{1/2}_{\pm 1})$ and $\mathrm{Tr}(\Lambda P^{1/2}_{\pm(1+\varepsilon)})$. Using (48) and (16) we obtain

$$\begin{aligned} \mathrm{Tr}(\Lambda P^{1/2}_{\pm 1}) &= \mp(4\varepsilon)^{-1}\,\mathrm{Tr}\{\Lambda[\beta_\varepsilon^3 - (1+2\varepsilon)\beta_\varepsilon]\}, \\ \mathrm{Tr}(\Lambda P^{1/2}_{\pm(1+\varepsilon)}) &= \pm(4\varepsilon)^{-1}(1+\varepsilon)\,\mathrm{Tr}[\Lambda(\beta_\varepsilon^3 - \beta_\varepsilon)]. \end{aligned} \tag{49}$$

From the expressions of β_ε and Λ, (30) and (31), and coefficients (47) we get

$$\mathrm{Tr}(\Lambda\beta_\varepsilon) = 4\varepsilon\rho_1, \qquad \mathrm{Tr}(\Lambda\beta_\varepsilon^3) = 12\varepsilon\rho_1.$$

Therefore

$$\begin{aligned} \gamma^1_{1/2} &= \pm\mathrm{Tr}(\Lambda P^{1/2}_{\pm 1}) = -2\rho_1(1-\varepsilon), \\ \gamma^2_{1/2} &= \pm\mathrm{Tr}(\Lambda P^{1/2}_{\pm(1+\varepsilon)}) = 2\rho_1(1+\varepsilon). \end{aligned} \tag{50}$$

If we remember that $\mathrm{sgn}\,\gamma$ gives us the sign of the charge density, we obtain from (50) that charge densities of different spin-$\frac{1}{2}$ particles are opposite (also in the limit $\varepsilon \to 0$). Therefore the fact that in the degenerate case charge density is indefinite (Loide and Loide 1977) is now explained: the charge densities of different $s = \frac{1}{2}$ states are opposite.

Examining the projection operators $P^{1/2}_{\pm 1}$ and $P^{1/2}_{\pm(1+\varepsilon)}$ in the limit $\varepsilon \to 0$ it is possible to verify by direct calculation that as $\varepsilon \to 0$ we get the new projection operators

$$\lim_{\varepsilon \to 0} P^{1/2}_{\pm 1} = P'_{\pm 1}, \qquad \lim_{\varepsilon \to 0} P^{1/2}_{\pm(1+\varepsilon)} = P''_{\pm 1},$$

which do not coincide with the projection operator $P_{\pm 1} = \pm\frac{1}{2}\beta(\beta \pm 1)$ of (41), but give

$$P_{\pm 1} = P'_{\pm 1} + P''_{\pm 1}.$$

This is not surprising, since $\psi^- = P'\psi$ and $\psi^+ = P''\psi$ give us solutions with definite charge density, but $P\psi$ gives the solution with indefinite charge density.

As we have seen, the prescription given in § 5 enables us to investigate the definiteness in the degenerate case. Therefore it appears that the algebraic definiteness conditions, based on the trace condition (12), are in principle applicable in the case of an arbitrary first-order wave equation.

8. Conclusions

In this paper we have analysed the known definiteness conditions and re-examined the cases when they are applicable. We have used the formalism based on the spin projection operators, which we find more convenient in algebraic investigations of equations.

As we have seen, the existing definiteness conditions work in the cases when there is no mass–spin degeneracy. For the degenerate case the ε procedure is given. Its idea is the following: we derive a new equation which has no mass–spin degencracy, and can therefore be tested for definiteness by the known trace conditions, and which in the limit $\varepsilon \to 0$ reduces to the equation we started with. The new equation and β_ε matrices may be useful in other calculations too because they allow us to separate different particles in the degenerate case.

Due to the acausality of high spin single-particle equations (Velo and Zwanziger 1969, Shamaly and Capri 1972) it becomes necessary to investigate different types of equations: single-particle and multi-particle equations. In the quantisation of multi-particle equations it is necessary to determine the definiteness of energy and charge. In definiteness testing the most useful technique is by using trace conditions, which as we have shown, may be regarded as the most universal definiteness conditions.

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